

# Some operators on interval-valued Hesitant fuzzy soft sets

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**ABSTRACT.** The main aim of this paper is to introduced the operations "Union" and "Intersection," and four operators  $O_1, O_2, O_3, O_4$  on interval-valued hesitant fuzzy soft sets and discuss some of their properties.

**Keywords:** Fuzzy soft sets; Interval-valued Hesitant fuzzy sets; Hesitant fuzzy soft sets.

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## 1. INTRODUCTION

Interval arithmetic was first suggested by Dwyer [8] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [17] in 1959 and Moore and Yang [18] in 1962. Further works on interval numbers can be found in Dwyer [9], Fischer [10]. Furthermore, Moore and Yang [19], have developed applications to differential equations. Chiao in [7] introduced sequence of interval numbers and defined usual convergence of sequences of interval number.

A set consisting of a closed interval of real numbers  $x$  such that  $a \leq x \leq b$  is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by  $\mathbf{IR}$ . Any elements of  $\mathbf{IR}$  is called closed interval and denoted by  $\bar{x}$ . That is  $\bar{x} = \{x \in \mathbb{R} : a \leq x \leq b\}$ . An interval number  $\bar{x}$  is a closed subset of real numbers [7]. Let  $x_l$  and  $x_r$  be first and last points of  $\bar{x}$  interval number, respectively. For  $\bar{x}_1, \bar{x}_2 \in \mathbf{IR}$ , we have  $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1l} = x_{2l}, x_{1r} = x_{2r}$ .  $\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R} : x_{1l} + x_{2l} \leq x \leq x_{1r} + x_{2r}\}$ , and if  $\alpha \geq 0$ , then  $\alpha\bar{x} = \{x \in \mathbb{R} : \alpha x_{1l} \leq x \leq \alpha x_{1r}\}$  and if  $\alpha < 0$ , then  $\alpha\bar{x} = \{x \in \mathbb{R} : \alpha x_{1r} \leq x \leq \alpha x_{1l}\}$ ,

$$\bar{x}_1 \cdot \bar{x}_2 = \left\{ x \in \mathbb{R} : \min \{x_{1l} \cdot x_{2l}, x_{1l} \cdot x_{2r}, x_{1r} \cdot x_{2l}, x_{1r} \cdot x_{2r}\} \leq x \leq \max \{x_{1l} \cdot x_{2l}, x_{1l} \cdot x_{2r}, x_{1r} \cdot x_{2l}, x_{1r} \cdot x_{2r}\} \right\}.$$

The Hesitant fuzzy set, as one of the extensions of Zadeh [29] fuzzy set, allows the membership degree that an element to a set presented by several possible values, and it can express the hesitant information more comprehensively than other extensions of fuzzy set. In 2009, Torra and Narukawa [22] introduced the concept of hesitant fuzzy set. In 2011, Xu and Xia [28] defined the concept of hesitant fuzzy element, which can be considered as the basic unit of a hesitant fuzzy set, and is a simple and effective tool used to express the decision makers hesitant preferences in the process of decision making. So many researchers has done lots of research work on aggregation, distance, similarity and correlation measures,

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clustering analysis, and decision making with hesitant fuzzy information. In 2013, Babitha and John [3] defined another important soft set Hesitant fuzzy soft sets. They introduced basic operations such as intersection, union, compliment and De Morgan's law was proved. In 2013, Chen et al. [6] extended hesitant fuzzy sets into interval-valued hesitant fuzzy environment and introduced the concept of interval-valued hesitant fuzzy sets. In 2015, Zhang et al. [30] introduced some operations such as complement, "AND", "OR", ring sum and ring product on interval-valued hesitant fuzzy soft sets.

There are many theories like theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of rough sets etc. which can be considered as mathematical tools for dealing with uncertain data, obtained in various fields of engineering, physics, computer science, economics, social science, medical science, and of many other diverse fields. But all these theories have their own difficulties. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets, introduced by L.A. Zadeh [29] in 1965. This theory brought a paradigmatic change in mathematics. But there exists difficulty, how to set the membership function in each particular case. The theory of intuitionistic fuzzy sets (see [1, 2]) is a more generalized concept than the theory of fuzzy sets, but this theory has the same difficulties. All the above mentioned theories are successful to some extent in dealing with problems arising due to vagueness present in the real world. But there are also cases where these theories failed to give satisfactory results, possibly due to inadequacy of the parameterization tool in them. As a necessary supplement to the existing mathematical tools for handling uncertainty, Molodtsov [15] introduced the theory of soft sets as a new mathematical tool to deal with uncertainties while modelling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. Molodtsov et al [16] successfully applied soft sets in directions such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, and theory of measurement. Maji et al [12] gave the first practical application of soft sets in decision-making problems. Maji et al [13] defined and studied several basic notions of the soft set theory. Also Çağman et al [5] studied several basic notions of the soft set theory. V. Torra [21, 22] and Verma and Sharma [23] discussed the relationship between hesitant fuzzy set and showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. Zhang et al [30] introduced weighted interval-valued hesitant fuzzy soft sets and finally applied it in decision making problem. Thakur et al [20] proposed four new operators  $O_1, O_2, O_3, O_4$  on hesitant fuzzy sets.

In this paper, in section 3, we study operations union and intersection on hesitant interval-valued fuzzy soft sets and some interesting properties of this notion. In section 4, we introduce four operators  $O_1, O_2, O_3, O_4$  in interval-valued hesitant fuzzy soft sets. Also various proposition are proved by using them.

## 2. PRELIMINARY RESULTS

In this section we recall some basic concepts and definitions regarding fuzzy soft sets, hesitant fuzzy set and hesitant fuzzy soft set.

**Definition 2.1.** [14] *Let  $U$  be an initial universe and  $F$  be a set of parameters. Let  $\tilde{P}(U)$  denote the power set of  $U$  and  $A$  be a non-empty subset of  $F$ . Then  $F_A$  is called a fuzzy soft set over  $U$  where  $F : A \rightarrow \tilde{P}(U)$  is a mapping from  $A$  into  $\tilde{P}(U)$ .*

**Definition 2.2.** [15]  *$F_E$  is called a soft set over  $U$  if and only if  $F$  is a mapping of  $E$  into the set of all subsets of the set  $U$ .*

In other words, the soft set is a parameterized family of subsets of the set  $U$ . Every set  $F(\epsilon)$ ,  $\epsilon \in E$ , from this family may be considered as the set of  $\epsilon$ -element of the soft set  $F_E$  or as the set of  $\epsilon$ -approximate elements of the soft set.

**Definition 2.3.** [2, 26] Let intuitionistic fuzzy value  $IFV(X)$  denote the family of all IFVs defined on the universe  $X$ , and let  $\alpha, \beta \in IFV(X)$  be given as:

$$\begin{aligned} \alpha &= (\mu_\alpha, \nu_\alpha), \beta = (\mu_\beta, \nu_\beta), \\ (i) \quad \alpha \cap \beta &= (\min(\mu_\alpha, \mu_\beta), \max(\nu_\alpha, \nu_\beta)) \\ (ii) \quad \alpha \cup \beta &= (\max(\mu_\alpha, \mu_\beta), \min(\nu_\alpha, \nu_\beta)) \\ (iii) \quad \alpha * \beta &= \left( \frac{\mu_\alpha + \mu_\beta}{2(\mu_\alpha \cdot \mu_\beta + 1)}, \frac{\nu_\alpha + \nu_\beta}{2(\nu_\alpha \cdot \nu_\beta + 1)} \right). \end{aligned}$$

**Definition 2.4.** [21] Given a fixed set  $X$ , then a hesitant fuzzy set (shortly HFS) in  $X$  is in terms of a function that when applied to  $X$  return a subset of  $[0, 1]$ . We express the HFS by a mathematical symbol:

$F = \{ \langle h, \mu_F(x) \rangle : h \in X \}$ , where  $\mu_F(x)$  is a set of some values in  $[0, 1]$ , denoting the possible membership degrees of the element  $h \in X$  to the set  $F$ .  $\mu_F(x)$  is called a hesitant fuzzy element (HFE) and  $H$  is the set of all HFEs.

**Definition 2.5.** [21] Given an hesitant fuzzy set  $F$ , define below it lower and upper bound as

$$\begin{aligned} \text{lower bound } F^-(x) &= \min F(x). \\ \text{upper bound } F^+(x) &= \max F(x). \end{aligned}$$

**Definition 2.6.** [21] Let  $\mu_1, \mu_2 \in H$  and three operations are defined as follows:

$$\begin{aligned} (1) \quad \mu_1^C &= \cup_{\gamma_1 \in \mu_1} \{1 - \gamma_1\}; \\ (2) \quad \mu_1 \cup \mu_2 &= \cup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \max\{\gamma_1, \gamma_2\}; \\ (3) \quad \mu_1 \cap \mu_2 &= \cap_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \min\{\gamma_1, \gamma_2\}. \end{aligned}$$

**Definition 2.7.** [6] Let  $X$  be a reference set, and  $D[0, 1]$  be the set of all closed subintervals of  $[0, 1]$ . An IVHFS on  $X$  is  $F = \{ \langle h_i, \mu_F(h_i) \rangle : h_i \in X, i = 1, 2, \dots, n \}$ , where  $\mu_F(h_i) : X \rightarrow D[0, 1]$  denotes all possible interval-valued membership degrees of the element  $h_i \in X$  to the set  $F$ . For convenience, we call  $\mu_F(h_i)$  an interval-valued hesitant fuzzy element (IVHFE), which reads  $\mu_F(h_i) = \{ \gamma : \gamma \in \mu_F(h_i) \}$ .

Here  $\gamma = [\gamma^L, \gamma^U]$  is an interval number.  $\gamma^L = \inf \gamma$  and  $\gamma^U = \sup \gamma$  represent the lower and upper limits of  $\gamma$ , respectively. An IVHFE is the basic unit of an IVHFS and it can be considered as a special case of the IVHFS. The relationship between IVHFE and IVHFS is similar to that between interval-valued fuzzy number and interval-valued fuzzy set.

**Example 2.8.** Let  $U = \{h_1, h_2\}$  be a reference set and let  $\mu_F(h_1) = \{[0.6, 0.8], [0.2, 0.7]\}$ ,  $\mu_F(h_2) = \{[0.1, 0.4]\}$  be the IVHFEs of  $h_i (i = 1, 2)$  to a set  $F$  respectively. Then IVHFS  $F$  can be written as  $F = \{ \langle h_1, \{[0.6, 0.8], [0.2, 0.7]\} \rangle, \langle h_2, \{[0.1, 0.4]\} \rangle \}$ .

**Definition 2.9.** [27] Let  $\tilde{a} = [\tilde{a}^L, \tilde{a}^U]$  and  $\tilde{b} = [\tilde{b}^L, \tilde{b}^U]$  be two interval numbers and  $\lambda \geq 0$ , then

$$\begin{aligned} (i) \quad \tilde{a} &= \tilde{b} \Leftrightarrow \tilde{a}^L = \tilde{b}^L \text{ and } \tilde{a}^U = \tilde{b}^U; \\ (ii) \quad \tilde{a} + \tilde{b} &= [\tilde{a}^L + \tilde{b}^L, \tilde{a}^U + \tilde{b}^U]; \\ (iii) \quad \lambda \tilde{a} &= [\lambda \tilde{a}^L, \lambda \tilde{a}^U], \text{ especially } \lambda \tilde{a} = 0, \text{ if } \lambda = 0. \end{aligned}$$

**Definition 2.10.** [27] Let  $\tilde{a} = [\tilde{a}^L, \tilde{a}^U]$  and  $\tilde{b} = [\tilde{b}^L, \tilde{b}^U]$ , and let  $l_a = \tilde{a}^U - \tilde{a}^L$  and  $l_b = \tilde{b}^U - \tilde{b}^L$ ; then the degree of possibility of  $\tilde{a} \geq \tilde{b}$  is formulated by

$$p(\tilde{a} \geq \tilde{b}) = \max\{1 - \max(\frac{\tilde{b}^U - \tilde{a}^L}{l_a + l_b}, 0), 0\}.$$

Above equation is proposed in order to compare two interval numbers, and to rank all the input arguments.

**Definition 2.11.** [6] For an IVHFE  $\tilde{\mu}$ ,  $s(\tilde{\mu}) = \frac{1}{l_{\tilde{\mu}}} \sum_{\tilde{\gamma} \in \tilde{\mu}} \tilde{\gamma}$  is called the score function of  $\tilde{\mu}$  with  $l_{\tilde{\mu}}$  being the number of the interval values in  $\tilde{\mu}$ , and  $s(\tilde{\mu})$  is an interval value belonging to  $[0, 1]$ . For two IVHFEs  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$ , if  $s(\tilde{\mu}_1) \geq s(\tilde{\mu}_2)$ , then  $\tilde{\mu}_1 \geq \tilde{\mu}_2$ .

We can judge the magnitude of two IVHFEs using above equation.

**Definition 2.12.** [6] Let  $\tilde{\mu}, \tilde{\mu}_1$  and  $\tilde{\mu}_2$  be three IVHFEs, then

- (i)  $\tilde{\mu}^C = \{[1 - \tilde{\gamma}^U, 1 - \tilde{\gamma}^L] : \tilde{\gamma} \in \tilde{\mu}\};$
- (ii)  $\tilde{\mu}_1 \cup \tilde{\mu}_2 = \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\};$
- (iii)  $\tilde{\mu}_1 \cap \tilde{\mu}_2 = \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\};$
- (iv)  $\tilde{\mu}_1 \oplus \tilde{\mu}_2 = \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\};$
- (v)  $\tilde{\mu}_1 \otimes \tilde{\mu}_2 = \{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] : \tilde{\gamma}_1 \in \tilde{\mu}_1, \tilde{\gamma}_2 \in \tilde{\mu}_2\}.$

**Proposition 2.13.** [6] For three IVHFEs  $\tilde{\mu}, \tilde{\mu}_1$  and  $\tilde{\mu}_2$ , we have

- (i)  $\tilde{\mu}_1^C \cup \tilde{\mu}_2^C = (\tilde{\mu}_1 \cap \tilde{\mu}_2)^C;$
- (ii)  $\tilde{\mu}_1^C \cap \tilde{\mu}_2^C = (\tilde{\mu}_1 \cup \tilde{\mu}_2)^C;$

**Definition 2.14.** [24] Let  $U$  be an initial universe and  $E$  be a set of parameters. Let  $\tilde{F}(U)$  be the set of all hesitant fuzzy subsets of  $U$ . Then  $F_E$  is called a hesitant fuzzy soft set (HFSS) over  $U$ , where  $\tilde{F} : E \rightarrow \tilde{F}(U)$ .

A HFSS is a parameterized family of hesitant fuzzy subsets of  $U$ , that is,  $\tilde{F}(U)$ . For all  $e \in E$ ,  $F(e)$  is referred to as the set of  $e$ - approximate elements of the HFSS  $F_E$ . It can be written as  $F(e) = \{< h, \mu_{F(e)(x)} > : h \in U\}$ .

Since HFE can represent the situation, in which different membership function are considered possible (see [21]),  $\mu_{F(e)(x)}$  is a set of several possible values, which is the hesitant fuzzy membership degree. In particular, if  $F(e)$  has only one element,  $F(e)$  can be called a hesitant fuzzy soft number. For convenience, a hesitant fuzzy soft number (HFSN) is denoted by  $\{< h, \mu_{F(e)(x)} >\}$ .

**Example 2.15.** Suppose  $U = \{h_1, h_2\}$  be an initial universe and  $E = \{e_1, e_2, e_3, e_4\}$  be a set of parameters. Let  $A = \{e_1, e_2\}$ . Then the hesitant fuzzy soft set  $F_A$  is given as  $F_A = \{F(e_1) = \{< h_1, \{0.6, 0.8\} >, < h_2, \{0.8, 0.4, 0.9\} >\}, F(e_2) = \{< h_1, \{0.9, 0.1, 0.5\} >, < h_2, \{0.2\} >\}$ .

**Definition 2.16.** [30] Let  $(U, E)$  be a soft universe and  $A \subseteq E$ . Then  $F_A$  is called an interval valued hesitant fuzzy soft set over  $U$ , where  $F$  is a mapping given by  $F : A \rightarrow IVHF(U)$ .

An interval-valued hesitant fuzzy soft set is a parameterized family of interval-valued hesitant fuzzy sub set of  $U$ . That is to say,  $F(e)$  is an interval-valued hesitant fuzzy subset in  $U, \forall e \in A$ . Following the standard notations,  $F(e)$  can be written as

$$F(e) = \{< h, \mu_{F(e)(x)} > : h \in U\}.$$

**Example 2.17.** Suppose  $U = \{h_1, h_2\}$  be an initial universe and  $E = \{e_1, e_2, e_3, e_4\}$  be a set of parameters. Let  $A = \{e_1, e_2\}$ . Then the interval valued hesitant fuzzy soft set  $F_A$  is given as

$$F_A = \{e_1 = \{< h_1, [0.6, 0.8] >, < h_2, [0.1, 0.4] >\}, \\ e_2 = \{< h_1, [0.2, 0.6], [0.3, 0.9] >, < h_2, [0.2, 0.5], [0.2, 0.8], [0.2, 0.8] >\}.$$

**Definition 2.18.** [30]  $U$  be an initial universe and let  $E$  be a set of parameters. Supposing that  $A, B \subseteq E$ ,  $F_A$  and  $F_B$  are two interval-valued hesitant fuzzy soft sets, one says that  $F_A$  is an interval-valued hesitant fuzzy soft subset of  $G_B$  if and only if

- (i)  $A \subseteq B$ ,
- (ii)  $\gamma_1^{\sigma(k)} \leq \gamma_2^{\sigma(k)}$ ,

where for all  $e \in A$ ,  $x \in U$ ,  $\gamma_1^{\sigma(k)}$  and  $\gamma_2^{\sigma(k)}$  stand for the  $k$ th largest interval number in the IVHFEs  $\mu_{F(e)(x)}$  and  $\mu_{G(e)(x)}$ , respectively. In this case, we write  $F_A \subseteq G_B$ .

**Definition 2.19.** [30] The complement of  $F_A$ , denoted by  $F_A^C$ , is defined by  $F_A^C(e) = \{< h, \mu_{F_A^C(e)(x)} > : h \in U\}$ , where  $\mu_{F_A^C} : A \rightarrow IVHF(U)$  is a mapping given by  $\mu_{F_A^C(e)}, \forall e \in A$  such that  $\mu_{F_A^C(e)}$  is the complement of interval-valued hesitant fuzzy element  $\mu_{F(e)}$  on  $U$ .

**Definition 2.20.** [30] An interval-valued hesitant fuzzy soft set is said to be an empty interval-valued hesitant fuzzy soft set, denoted by  $\tilde{\phi}$ , if  $F : E \rightarrow IVHF(U)$  such that  $F(\tilde{e}) = \{ \langle h, \mu_{F(\tilde{e})}(x) \rangle : h \in U \} = \{ \langle h, \{[0, 0]\} \rangle : h \in U \}, \forall \tilde{e} \in E$ .

**Definition 2.21.** [30] An interval-valued hesitant fuzzy soft set is said to be an full interval-valued hesitant fuzzy soft set, denoted by  $\tilde{E}$ , if  $F : E \rightarrow IVHF(U)$  such that  $F(\tilde{e}) = \{ \langle h, \mu_{F(\tilde{e})}(x) \rangle : h \in U \} = \{ \langle h, \{[1, 1]\} \rangle : h \in U \}, \forall \tilde{e} \in E$ .

**Definition 2.22.** [30] The ring sum operation on the two interval-valued hesitant fuzzy soft sets  $F_A, G_B$  over  $(U, E)$ , denoted by  $F_A \oplus G_A = H$ , is a mapping given by  $H : E \rightarrow IVHF(U)$  such that

$$\forall \tilde{e} \in E \quad H(\tilde{e}) = \{ \langle h, \mu_{H(\tilde{e})}(x) \rangle : h \in U \} = \{ \langle h, \mu_{F(\tilde{e})}(x) \oplus \mu_{G(\tilde{e})}(x) \rangle : h \in U \}, \forall \tilde{e} \in E.$$

**Definition 2.23.** [30] The ring product operation on the two interval-valued hesitant fuzzy soft sets  $F_A, G_B$  over  $(U, E)$ , denoted by  $F_A \otimes G_A = H$ , is a mapping given by  $H : E \rightarrow IVHF(U)$  such that

$$\forall \tilde{e} \in E \quad H(\tilde{e}) = \{ \langle h, \mu_{H(\tilde{e})}(x) \rangle : h \in U \} = \{ \langle h, \mu_{F(\tilde{e})}(x) \otimes \mu_{G(\tilde{e})}(x) \rangle : h \in U \}, \forall \tilde{e} \in E.$$

### 3. MAIN RESULTS

**Definition 3.1.** The union of two interval-valued hesitant fuzzy soft sets  $F_A$  and  $G_B$  over  $(U, E)$ , is the interval-valued hesitant fuzzy soft set  $H_C$ , where  $C = A \cup B$  and  $\forall \tilde{e} \in C$ ,

$$\mu_{H(e)} = \begin{cases} \mu_{F(e)}, & \text{if } \tilde{e} \in A - B; \\ \mu_{G(e)}, & \text{if } \tilde{e} \in B - A; \\ \mu_{F(e)} \cup \mu_{G(e)}, & \text{if } \tilde{e} \in A \cap B. \end{cases}$$

We write  $F_A \tilde{\cup} G_B = H_C$ .

**Example 3.2.** Let  $F_A = \{e_1 = \{ \langle h_1, [0.3, 0.8] \rangle, \langle h_2, [0.3, 0.8], [0.5, 0.6], [0.3, 0.6] \rangle \}$   
 $e_2 = \{ \langle h_1, [0.2, 0.9], [0.7, 1.0] \rangle, \langle h_2, [0.8, 1.0], [0.2, 0.6] \rangle \}$ .

$$\begin{aligned} G_B &= \{e_1 = \{ \langle h_1, [0.7, 0.9], [0.0, 0.6] \rangle, \langle h_2, [0.4, 0.7], [0.4, 0.5] \rangle \} \\ e_2 &= \{ \langle h_1, [0.6, 0.8] \rangle, \langle h_2, [0.3, 0.8], [0.3, 0.6] \rangle \} \\ e_3 &= \{ \langle h_1, [0.5, 0.6], [0.3, 0.6] \rangle, \langle h_2, [0.1, 0.6], [0.3, 0.9], [0.3, 0.6] \rangle \}. \end{aligned}$$

Now rearrange the membership value of  $F_A$  and  $G_B$  with the help of Definitions 2.9, 2.10 and assumptions given by [6], we have

$$\begin{aligned} F_A &= \{e_1 = \{ \langle h_1, [0.3, 0.8] \rangle, \langle h_2, [0.3, 0.6], [0.3, 0.8], [0.5, 0.6] \rangle \} \\ e_2 &= \{ \langle h_1, [0.2, 0.9], [0.7, 1.0] \rangle, \langle h_2, [0.2, 0.6], [0.8, 1.0] \rangle \}. \end{aligned}$$

$$\begin{aligned} G_B &= \{e_1 = \{ \langle h_1, [0.0, 0.6], [0.7, 0.9] \rangle, \langle h_2, [0.4, 0.5], [0.4, 0.7], [0.4, 0.7] \rangle \} \\ e_2 &= \{ \langle h_1, [0.6, 0.8] \rangle, \langle h_2, [0.3, 0.6], [0.3, 0.8] \rangle \} \\ e_3 &= \{ \langle h_1, [0.3, 0.6], [0.5, 0.6] \rangle, \langle h_2, [0.1, 0.6], [0.3, 0.9], [0.3, 0.6] \rangle \}. \end{aligned}$$

Therefore

$$\begin{aligned} F_A \tilde{\cup} G_B &= H_{A \cup B} = H_C \\ &= \{e_1 = \{ \langle h_1, [0.3, 0.8], [0.7, 0.9] \rangle, \langle h_2, [0.4, 0.6], [0.4, 0.8], [0.5, 0.7] \rangle \} \\ e_2 &= \{ \langle h_1, [0.6, 0.9], [0.7, 1.0] \rangle, \langle h_2, [0.3, 0.6], [0.8, 1.0] \rangle \} \\ e_3 &= \{ \langle h_1, [0.3, 0.6], [0.5, 0.6] \rangle, \langle h_2, [0.1, 0.6], [0.3, 0.9], [0.3, 0.6] \rangle \}. \end{aligned}$$

**Definition 3.3.** The intersection of two interval-valued hesitant fuzzy soft sets  $F_A$  and  $G_B$  with  $A \cap B \neq \phi$  over  $(U, E)$ , is the interval-valued hesitant fuzzy soft set  $H_C$ , where  $C = A \cap B$ , and  $\forall \tilde{e} \in C, \mu_{H(e)} = \mu_{F(e)} \cap \mu_{G(e)}$ . We write  $F_A \tilde{\cap} G_B = H_C$ .

**Example 3.4.** From Example 3.2, we have

$$\begin{aligned} F_A \tilde{\cap} G_B &= H_A \tilde{\cap} B = H_C \\ &= \{e_1 = \{ \langle h_1, [0.0, 0.6], [0.3, 0.8] \rangle, \langle h_2, [0.3, 0.5], [0.3, 0.7] \rangle [0.4, 0.6] \rangle \} \\ e_2 &= \{ \langle h_1, [0.2, 0.8] [0.6, 0.8] \rangle, \langle h_2, [0.2, 0.6], [0.3, 0.8] \rangle \}. \end{aligned}$$

**Proposition 3.5.** Let  $F_A$  be a interval-valued hesitant fuzzy soft set. Then the following are true:

- (i)  $F_A \tilde{\cup} F_A = F_A$
- (ii)  $F_A \tilde{\cap} F_A = F_A$
- (iii)  $F_A \tilde{\cup} \tilde{\phi}_A = F_A$
- (iv)  $F_A \tilde{\cap} \tilde{\phi}_A = \tilde{\phi}_A$
- (v)  $F_A \tilde{\cup} \tilde{E}_A = \tilde{E}_A$
- (vi)  $F_A \tilde{\cap} \tilde{E}_A = F_A$ .

*Proof.* Obvious. □

**Proposition 3.6.** Let  $F_A$  and  $G_A$  are two interval-valued hesitant fuzzy soft sets. Then

- (i)  $(F_A \tilde{\cup} G_A)^C = F_A^C \tilde{\cap} G_A^C$
- (ii)  $(F_A \tilde{\cap} G_A)^C = F_A^C \tilde{\cup} G_A^C$ .

*Proof.* (i) Let  $F_A^C \tilde{\cap} G_A^C = H_A$ .

We have  $\forall e \in A, \mu_{H(e)} = \mu_{F^C(e)} \cap \mu_{G^C(e)}$ . .....(A1)

Suppose that  $F_A \tilde{\cup} G_A = L_A$

Therefore,  $(F_A \tilde{\cup} G_A)^C = L_A^C$ .

We have  $\forall e \in A, \mu_{L^C(e)} = (\mu_{F(e)} \cup \mu_{G(e)})^C = \mu_{F^C(e)} \cap \mu_{G^C(e)}$ . .....(A2)

From (A1) and (A2),  $(F_A \tilde{\cup} G_A)^C = F_A^C \tilde{\cap} G_A^C$ .

(ii) Let  $F_A^C \tilde{\cup} G_A^C = P_A$ .

We have  $\forall e \in A, \mu_{P(e)} = \mu_{F^C(e)} \cup \mu_{G^C(e)}$ . .....(B1)

Suppose that  $F_A \tilde{\cap} G_A = Q_A$

Therefore  $(F_A \tilde{\cap} G_A)^C = Q_A^C$ .

We have  $\forall e \in A, \mu_{Q^C(e)} = (\mu_{F(e)} \cap \mu_{G(e)})^C = \mu_{F^C(e)} \cup \mu_{G^C(e)}$ . .....(B2)

From (B1) and (B2),  $(F_A \tilde{\cap} G_A)^C = F_A^C \tilde{\cup} G_A^C$ . □

**Proposition 3.7.** Let  $F_A$  and  $G_B$  are two interval-valued hesitant fuzzy soft sets. Then the following are satisfied:

- (i)  $F_A^C \tilde{\cap} G_B^C \subseteq (F_A \tilde{\cup} G_B)^C$
- (ii)  $(F_A \tilde{\cap} G_B)^C \subseteq F_A^C \tilde{\cup} G_B^C$
- (iii)  $F_A^C \tilde{\cap} G_B^C \subseteq (F_A \tilde{\cap} G_B)^C$
- (iv)  $(F_A \tilde{\cup} G_B)^C \subseteq F_A^C \tilde{\cup} G_B^C$ .

*Proof.* From Example 3.2

$$\begin{aligned} (i) \quad (F_A \tilde{\cup} G_B)^C &= \{e_1 = \{ \langle h_1, [0.1, 0.3], [0.2, 0.7] \rangle, \langle h_2, [0.3, 0.5], [0.2, 0.6] \rangle [0.4, 0.6] \rangle \} \\ e_2 &= \{ \langle h_1, [0.0, 0.3] [0.1, 0.4] \rangle, \langle h_2, [0.0, 0.2], [0.4, 0.7] \rangle \} \\ e_3 &= \{ \langle h_1, [0.4, 0.5], [0.4, 0.7] \rangle, \langle h_2, [0.4, 0.7], [0.1, 0.7], [0.4, 0.9] \rangle \}. \end{aligned}$$

$$\begin{aligned} F_A^C &= \{e_1 = \{ \langle h_1, [0.2, 0.7] \rangle, \langle h_2, [0.4, 0.5], [0.2, 0.7], [0.4, 0.7] \rangle \} \\ e_2 &= \{ \langle h_1, [0.0, 0.3], [0.1, 0.8] \rangle, \langle h_2, [0.0, 0.2], [0.4, 0.8] \rangle \}. \end{aligned}$$

$$\begin{aligned} G_B^C &= \{e_1 = \{ \langle h_1, [0.1, 0.3], [0.4, 1.0] \rangle, \langle h_2, [0.3, 0.6], [0.3, 0.6], [0.5, 0.6] \rangle \} \\ e_2 &= \{ \langle h_1, [0.2, 0.4] \rangle, \langle h_2, [0.2, 0.7], [0.4, 0.7] \rangle \} \end{aligned}$$

$$e_3 = \{ \langle h_1, [0.4, 0.5], [0.4, 0.7] \rangle, \langle h_2, [0.4, 0.7], [0.1, 0.7], [0.4, 0.9] \rangle \}.$$

$$\begin{aligned} F_A^C \tilde{\cap} G_B^C &= \{e_1 = \{ \langle h_1, [0.1, 0.3], [0.2, 0.7] \rangle, \langle h_2, [0.3, 0.5], [0.2, 0.6], [0.4, 0.6] \rangle \} \\ e_2 &= \{ \langle h_1, [0.0, 0.3], [0.1, 0.4] \rangle, \langle h_2, [0.0, 0.2], [0.4, 0.7] \rangle \}. \end{aligned}$$

$$\text{Hence } F_A^C \tilde{\cap} G_B^C \subseteq (F_A \tilde{\cup} G_B)^C.$$

$$\begin{aligned} \text{(ii) } (F_A \tilde{\cap} G_B)^C &= \{e_1 = \{ \langle h_1, [0.2, 0.7], [0.4, 1.0] \rangle, \langle h_2, [0.4, 0.6], [0.3, 0.7], [0.5, 0.7] \rangle \} \\ e_2 &= \{ \langle h_1, [0.2, 0.4], [0.2, 0.8] \rangle, \langle h_2, [0.2, 0.7], [0.4, 0.8] \rangle \}. \end{aligned}$$

$$\begin{aligned} F_A^C \tilde{\cup} G_B^C &= \{e_1 = \{ \langle h_1, [0.2, 0.7], [0.4, 1.0] \rangle, \langle h_2, [0.4, 0.6], [0.3, 0.7], [0.5, 0.7] \rangle \} \\ e_2 &= \{ \langle h_1, [0.2, 0.4], [0.2, 0.8] \rangle, \langle h_2, [0.2, 0.7], [0.4, 0.8] \rangle \} \\ e_3 &= \{ \langle h_1, [0.4, 0.5], [0.4, 0.7] \rangle, \langle h_2, [0.4, 0.7], [0.1, 0.7], [0.4, 0.9] \rangle \}. \\ \text{Hence } (F_A \tilde{\cap} G_B)^C &\subseteq F_A^C \tilde{\cup} G_B^C. \end{aligned}$$

(iii) From (i) and (ii) we get the result.

(iv) From (i) and (ii) we get the result.  $\square$

**Proposition 3.8.** Let  $F_A, G_B$  and  $H_C$  are three interval-valued hesitant fuzzy soft sets. Then the following are satisfied:

- (i)  $F_A \tilde{\cup} G_B = G_B \tilde{\cup} F_A$
- (ii)  $F_A \tilde{\cap} G_B = G_B \tilde{\cap} F_A$
- (iii)  $F_A \tilde{\cup} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cup} H_C$
- (iv)  $F_A \tilde{\cap} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cap} H_C$ .

*Proof.* The proof can be obtained from definition 3.1 and definition 3.3.  $\square$

**Proposition 3.9.** Let  $F_A, G_A$  and  $H_A$  are three interval-valued hesitant fuzzy soft sets. Then the following propositiones are satiesfied:

- (i)  $F_A \tilde{\cup} G_A = G_A \tilde{\cup} F_A$
- (ii)  $F_A \tilde{\cap} G_A = G_A \tilde{\cap} F_A$
- (iii)  $F_A \tilde{\cup} (G_A \tilde{\cup} H_A) = (F_A \tilde{\cup} G_A) \tilde{\cup} H_A$
- (iv)  $F_A \tilde{\cap} (G_A \tilde{\cap} H_A) = (F_A \tilde{\cap} G_A) \tilde{\cap} H_A$ .

*Proof.* The proof can be obtained from definition 3.1 and definition 3.3.  $\square$

**Proposition 3.10.** Let  $F_A, G_A$  and  $H_A$  are three interval-valued hesitant fuzzy soft sets. Then the following are satisfied:

- (i)  $F_A \tilde{\cup} (G_A \tilde{\cap} H_A) = (F_A \tilde{\cup} G_A) \tilde{\cap} (F_A \tilde{\cup} H_A)$
- (ii)  $F_A \tilde{\cap} (G_A \tilde{\cup} H_A) = (F_A \tilde{\cap} G_A) \tilde{\cup} (F_A \tilde{\cap} H_A)$ .

*Proof.* Obvious.  $\square$

**Proposition 3.11.** Let  $F_A, G_B$  and  $H_C$  are three interval-valued hesitant fuzzy soft sets. Then the following are not satisfied:

- (i)  $F_A \tilde{\cup} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cap} (F_A \tilde{\cup} H_C)$
- (ii)  $F_A \tilde{\cap} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cup} (F_A \tilde{\cap} H_C)$ .

*Proof.* We consider a example.

Let  $H_C = \{e_2 = \{ \langle h_1, [0.4, 0.6], [0.2, 0.6], [0.7, 1.0] \rangle, \langle h_2, [0.3, 0.8], > \} \\ e_3 = \{ \langle h_1, [0.2, 0.5], [0.3, 0.5] \rangle, \langle h_2, [0.6, 0.8], [0.2, 0.5] \rangle \}.$   
Now by Definitions 2.9 and 2.10 and assumptions given by [6]  
 $H_C = \{e_2 = \{ \langle h_1, [0.2, 0.6], [0.4, 0.6], [0.7, 1.0] \rangle, \langle h_2, [0.3, 0.8], > \} \\ e_3 = \{ \langle h_1, [0.2, 0.5], [0.3, 0.5] \rangle, \langle h_2, [0.2, 0.5], [0.6, 0.8] \rangle \}.$

(i) From example 3.2, we have

$$\begin{aligned} F_A \tilde{\cup} H_C &= \{e_1 = \{< h_1, [0.3, 0.8] >, < h_2, [0.3, 0.6], [0.3, 0.8][0.5, 0.6] >\} \\ e_2 &= \{< h_1, [0.2, 0.9], [0.7, 1.0], [0.7, 1.0] >, < h_2, [0.3, 0.8], [0.8, 1.0] >\} \\ e_3 &= \{< h_1, [0.2, 0.5], [0.3, 0.5] >, < h_2, [0.2, 0.5], [0.6, 0.8] >\}. \\ (F_A \tilde{\cup} G_B) \tilde{\cap} (F_A \tilde{\cup} H_C) \\ &= \{e_1 = \{< h_1, [0.3, 0.8], [0.3, 0.8] >, < h_2, [0.3, 0.6], [0.3, 0.8][0.5, 0.6] >\} \\ e_2 &= \{< h_1, [0.2, 0.9], [0.7, 1.0], [0.7, 1.0] >, < h_2, [0.3, 0.6], [0.8, 1.0] >\} \\ e_3 &= \{< h_1, [0.2, 0.5], [0.3, 0.5] >, < h_2, [0.1, 0.5], [0.3, 0.8], [0.3, 0.6] >\}. \end{aligned}$$

Again

$$\begin{aligned} G_B \tilde{\cap} H_C &= \{e_2 = \{< h_1, [0.2, 0.6], [0.4, 0.6], [0.6, 0.8] >, < h_2, [0.3, 0.6], [0.3, 0.8] >\} \\ e_3 &= \{< h_1, [0.2, 0.5], [0.3, 0.5] >, < h_2, [0.1, 0.5], [0.3, 0.8], [0.3, 0.6] >\}. \end{aligned}$$

Therefore

$$\begin{aligned} F_A \tilde{\cup} (G_B \tilde{\cap} H_C) \\ &= \{e_1 = \{< h_1, [0.3, 0.8] >, < h_2, [0.3, 0.6], [0.3, 0.8][0.5, 0.6] >\} \\ e_2 &= \{< h_1, [0.2, 0.9], [0.7, 1.0], [0.7, 1.0] >, < h_2, [0.3, 0.6], [0.8, 1.0] >\} \\ e_3 &= \{< h_1, [0.2, 0.5], [0.3, 0.5] >, < h_2, [0.1, 0.5], [0.3, 0.8], [0.3, 0.6] >\}. \\ \text{Hence } F_A \tilde{\cup} (G_B \tilde{\cap} H_C) &\neq (F_A \tilde{\cup} G_B) \tilde{\cap} (F_A \tilde{\cup} H_C). \end{aligned}$$

(ii) From example 3.2 and 3.4.

$$\begin{aligned} G_B \tilde{\cup} H_C &= \{e_1 = \{< h_1, [0.0, 0.6], [0.7, 0.9] >, < h_2, [0.4, 0.5], [0.4, 0.7][0.4, 0.7] >\} \\ e_2 &= \{< h_1, [0.6, 0.8], [0.6, 0.8], [0.7, 1.0] >, < h_2, [0.3, 0.8], [0.3, 0.8] >\} \\ e_3 &= \{< h_1, [0.3, 0.6], [0.5, 0.6] >, < h_2, [0.2, 0.6], [0.6, 0.9], [0.6, 0.8] >\}. \end{aligned}$$

Therefore

$$\begin{aligned} F_A \tilde{\cap} (G_B \tilde{\cup} H_C) \\ &= \{e_1 = \{< h_1, [0.0, 0.6], [0.3, 0.8] >, < h_2, [0.3, 0.5], [0.3, 0.7][0.4, 0.6] >\} \\ e_2 &= \{< h_1, [0.2, 0.8], [0.6, 0.8], [0.7, 1.0] >, < h_2, [0.2, 0.6], [0.3, 0.8] >\}. \end{aligned}$$

Again

$$F_A \tilde{\cap} H_C = \{e_2 = \{< h_1, [0.2, 0.6], [0.4, 0.6], [0.7, 1.0] >, < h_2, [0.2, 0.6], [0.3, 0.8] >\}.$$

Therefore

$$\begin{aligned} (F_A \tilde{\cap} G_B) \tilde{\cup} (F_A \tilde{\cap} H_C) \\ &= \{e_2 = \{< h_1, [0.2, 0.8], [0.6, 0.8], [0.7, 1.0] >, < h_2, [0.2, 0.6], [0.3, 0.8] >\}. \end{aligned}$$

$$\text{Hence } F_A \tilde{\cap} (G_B \tilde{\cup} H_C) \neq (F_A \tilde{\cap} G_B) \tilde{\cup} (F_A \tilde{\cap} H_C). \quad \square$$

**Definition 3.12.** Let  $\mathfrak{R} = \{(F_i)_{A_i} : i \in I\}$  be a family of hesitant fuzzy soft sets over  $(U, E)$ . Then the union of hesitant fuzzy soft sets in  $\mathfrak{R}$  is a hesitant fuzzy soft set  $H_K, K = \cup_i A_i$  and  $\forall e \in E, K(e) = \cup_i (\Delta_i)_{A_i}(e)$ , where

$$(\Delta_i)_{A_i}(e) = \begin{cases} F_i(e), & \text{if } e \in A_i \\ \phi, & \text{if } e \notin A_i. \end{cases}$$

**Example 3.13.** Let  $(F_1)_{A_1} = \{e_1 = \{< h_1, [0.3, 0.8] >, < h_2, [0.3, 0.8], [0.5, 0.6], [0.3, 0.6] >\} \\ e_2 = \{< h_1, [0.2, 0.9], [0.7, 1.0] >, < h_2, [0.8, 1.0], [0.2, 0.6] >\}.$

$$\begin{aligned} (F_2)_{A_2} &= \{e_1 = \{< h_1, [0.7, 0.9], [0.0, 0.6] >, < h_2, [0.4, 0.7], [0.4, 0.5] >\} \\ e_2 &= \{< h_1, [0.6, 0.8] >, < h_2, [0.3, 0.8], [0.3, 0.6] >\} \\ e_3 &= \{< h_1, [0.5, 0.6], [0.3, 0.6] >, < h_2, [0.1, 0.6], [0.3, 0.9], [0.3, 0.6] >\}. \\ (F_3)_{A_3} &= \{e_2 = \{< h_1, [0.4, 0.6], [0.2, 0.6], [0.7, 1.0] >, < h_2, [0.3, 0.8], >\} \\ e_3 &= \{< h_1, [0.2, 0.5], [0.3, 0.5] >, < h_2, [0.6, 0.8], [0.2, 0.5] >\}. \end{aligned}$$

Therefore

$$\begin{aligned} (F_1)_{A_1} \tilde{\cup} (F_2)_{A_2} \tilde{\cup} (F_3)_{A_3} \\ &= \{e_1 = \{< h_1, [0.3, 0.8], [0.7, 0.9] >, < h_2, [0.4, 0.6], [0.4, 0.8], [0.5, 0.7] >\} \end{aligned}$$



$$e_2 = \{ \langle h_1, [0.6, 0.9], [0.7, 1.0], [0.7, 1.0] \rangle, \langle h_2, [0.3, 0.8], [0.8, 1.0] \rangle \}$$

$$e_3 = \{ \langle h_1, [0.3, 0.6], [0.5, 0.6] \rangle, \langle h_2, [0.2, 0.6], [0.6, 0.9], [0.6, 0.8] \rangle \}.$$

**Definition 3.14.** Let  $\mathfrak{R} = \{(F_i)_{A_i} : i \in I\}$  be a family of hesitant fuzzy soft sets with  $\cap_i A_i \neq \emptyset$  over  $(U, E)$ . Then the intersection of hesitant fuzzy soft sets in  $\mathfrak{R}$  is a hesitant fuzzy soft set  $H_K, K = \cap_i A_i$  and  $\forall e \in E, K(e) = \cap_i A_i(e)$ .

**Example 3.15.** From Example 3.13, we have

$$(F_1)_{A_1} \tilde{\cap} (F_2)_{A_2} \tilde{\cap} (F_3)_{A_3}$$

$$= \{e_2 = \{ \langle h_1, [0.2, 0.6], [0.4, 0.6], [0.6, 0.8] \rangle, \langle h_2, [0.2, 0.6], [0.3, 0.8] \rangle \}.$$

**Proposition 3.16.** Let  $\mathfrak{R} = \{(F_i)_{A_i} : i \in I\}$  be a family of hesitant fuzzy soft sets over  $(U, E)$ . Then

- (i)  $\tilde{\bigcap}_i (F_i)_{A_i} \overset{C}{\subseteq} (\tilde{\bigcup}_i (F_i)_{A_i})^C$
- (ii)  $(\tilde{\bigcap}_i (F_i)_{A_i})^C \subseteq \bigcup_i (F_i)_{A_i}^C$ .

*Proof.* Obvious. □

**Proposition 3.17.** Let  $\mathfrak{R} = \{(F_i)_A : i \in I\}$  be a family of hesitant fuzzy soft sets over  $(U, E)$ . Then

- (i)  $\tilde{\bigcap}_i (F_i)_A \overset{C}{=} (\tilde{\bigcup}_i (F_i)_A)^C$
- (ii)  $(\tilde{\bigcap}_i (F_i)_A)^C = \bigcup_i (F_i)_A^C$ .

*Proof.* Obvious. □

#### 4. NEW OPERATORS ON INTERVAL-VALUED HESITANT FUZZY SOFT ELEMENTS

**Definition 4.1.** Let  $\tilde{\mu}_1, \tilde{\mu}_2$  be two interval-valued hesitant fuzzy soft elements (IVHFSEs) of same set of parameters, then

- (i)  $\tilde{\mu}_1 O_1 \tilde{\mu}_2 = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \left[ \frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|} \right]$
- (ii)  $\tilde{\mu}_1 O_2 \tilde{\mu}_2 = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \left[ \frac{|\gamma_1^L - \gamma_2^L|}{1 + 2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + 2|\gamma_1^U - \gamma_2^U|} \right]$
- (iii)  $\tilde{\mu}_1 O_3 \tilde{\mu}_2 = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \left[ \frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2} \right]$
- (iv)  $\tilde{\mu}_1 O_4 \tilde{\mu}_2 = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} \left[ \frac{|\gamma_1^L * \gamma_2^L|}{2}, \frac{|\gamma_1^U * \gamma_2^U|}{2} \right].$

**Proposition 4.2.** If  $\tilde{\mu}_1, \tilde{\mu}_2$  and  $\tilde{\mu}_3$  be two interval-valued hesitant fuzzy soft elements. Then the following identities are true:

- (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_1 \tilde{\mu}_2) = \tilde{\mu}_1 O_1 \tilde{\mu}_2,$
- (ii)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_1 \tilde{\mu}_2) = \tilde{\mu}_1 \oplus \tilde{\mu}_2,$
- (iii)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_1 \tilde{\mu}_2) = \tilde{\mu}_1 O_1 \tilde{\mu}_2,$
- (iv)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_1 \tilde{\mu}_2) = \tilde{\mu}_1 \otimes \tilde{\mu}_2,$
- (v)  $(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_1 \tilde{\mu}_3 = (\tilde{\mu}_1 O_1 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_1 \tilde{\mu}_3),$
- (vi)  $(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_1 \tilde{\mu}_3 = (\tilde{\mu}_1 O_1 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_1 \tilde{\mu}_3),$

*Proof.* (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_1 \tilde{\mu}_2)$

$$= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}])$$

$$= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}\}, \min\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}\}]$$

$$= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}]$$

$$= \tilde{\mu}_1 O_1 \tilde{\mu}_2.$$

(ii)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_1 \tilde{\mu}_2)$

$$= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}])$$

$$= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}\}, \max\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}\}]$$

$$= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \\ = \tilde{\mu}_1 \oplus \tilde{\mu}_2.$$

$$(iii) (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_1 \tilde{\mu}_2) \\ = (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}]) \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}\}, \min\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}\}] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}] \\ = \tilde{\mu}_1 O_1 \tilde{\mu}_2.$$

$$(iv) (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_1 \tilde{\mu}_2) \\ = (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}]) \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}\}, \max\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}\}] \\ = (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \\ = \tilde{\mu}_1 \otimes \tilde{\mu}_2.$$

$$(v) (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_1 \tilde{\mu}_3 \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \max\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_1 \bigcup_{\gamma_3 \in \mu_3} [\tilde{\gamma}_3^L, \tilde{\gamma}_3^U] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\frac{|\max\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}{1 + |\max\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}, \frac{|\max\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}{1 + |\max\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\max\{\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_2^L - \gamma_3^L|}{1 + |\gamma_2^L - \gamma_3^L|}\}, \max\{\frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}, \frac{|\gamma_2^U - \gamma_3^U|}{1 + |\gamma_2^U - \gamma_3^U|}\}] \\ = (\tilde{\mu}_1 O_1 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_1 \tilde{\mu}_3).$$

$$(vi) (\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_1 \tilde{\mu}_3 \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \min\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_1 \bigcup_{\gamma_3 \in \mu_3} [\tilde{\gamma}_3^L, \tilde{\gamma}_3^U] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\frac{|\min\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}{1 + |\min\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}, \frac{|\min\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}{1 + |\min\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\frac{|\gamma_1^L - \gamma_2^L|}{1 + |\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_2^L - \gamma_3^L|}{1 + |\gamma_2^L - \gamma_3^L|}\}, \min\{\frac{|\gamma_1^U - \gamma_2^U|}{1 + |\gamma_1^U - \gamma_2^U|}, \frac{|\gamma_2^U - \gamma_3^U|}{1 + |\gamma_2^U - \gamma_3^U|}\}] \\ = (\tilde{\mu}_1 O_1 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_1 \tilde{\mu}_3). \quad \square$$

**Proposition 4.3.** If  $\tilde{\mu}_1, \tilde{\mu}_2$  and  $\tilde{\mu}_3$  be two interval-valued hesitant fuzzy soft elements. Then the following identities are true:

- (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) = \tilde{\mu}_1 O_2 \tilde{\mu}_2,$
- (ii)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) = \tilde{\mu}_1 \oplus \tilde{\mu}_2,$
- (iii)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) = \tilde{\mu}_1 O_2 \tilde{\mu}_2,$
- (iv)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) = \tilde{\mu}_1 \otimes \tilde{\mu}_2,$
- (v)  $(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_2 \tilde{\mu}_3 = (\tilde{\mu}_1 O_2 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_2 \tilde{\mu}_3),$
- (vi)  $(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_2 \tilde{\mu}_3 = (\tilde{\mu}_1 O_2 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_2 \tilde{\mu}_3).$

*Proof.* (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_2 \tilde{\mu}_2)$

$$= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + 2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + 2|\gamma_1^U - \gamma_2^U|}]) \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1 + 2|\gamma_1^L - \gamma_2^L|}\}, \min\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1 + 2|\gamma_1^U - \gamma_2^U|}\}] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + 2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + 2|\gamma_1^U - \gamma_2^U|}] \\ = \tilde{\mu}_1 O_2 \tilde{\mu}_2.$$

$$(ii) (\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) \\ = (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1 + 2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1 + 2|\gamma_1^U - \gamma_2^U|}]) \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1 + 2|\gamma_1^L - \gamma_2^L|}\}, \max\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1 + 2|\gamma_1^U - \gamma_2^U|}\}] \\ = \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]$$

$$= \tilde{\mu}_1 \oplus \tilde{\mu}_2.$$

$$\begin{aligned} & \text{(iii)} \quad (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) \\ &= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}]) \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}\}, \min\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}\}] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}] \\ &= \tilde{\mu}_1 O_2 \tilde{\mu}_2. \end{aligned}$$

$$\begin{aligned} & \text{(iv)} \quad (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_2 \tilde{\mu}_2) \\ &= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}]) \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}\}, \max\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}\}] \\ &= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \\ &= \tilde{\mu}_1 \otimes \tilde{\mu}_2. \end{aligned}$$

$$\begin{aligned} & \text{(v)} \quad (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_2 \tilde{\mu}_3 \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \max\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_2 \bigcup_{\gamma_3 \in \mu_3} [\tilde{\gamma}_3^L, \gamma_3^U] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\frac{|\max\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}{1+2|\max\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}, \frac{|\max\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}{1+2|\max\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_2^L - \gamma_3^L|}{1+2|\gamma_2^L - \gamma_3^L|}\}, \max\{\frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}, \frac{|\gamma_2^U - \gamma_3^U|}{1+2|\gamma_2^U - \gamma_3^U|}\}] \\ &= (\tilde{\mu}_1 O_2 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_2 \tilde{\mu}_3). \end{aligned}$$

$$\begin{aligned} & \text{(vi)} \quad (\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_2 \tilde{\mu}_3 \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \min\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_2 \bigcup_{\gamma_3 \in \mu_3} [\tilde{\gamma}_3^L, \gamma_3^U] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\frac{|\min\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}{1+2|\min\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}, \frac{|\min\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}{1+2|\min\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\frac{|\gamma_1^L - \gamma_2^L|}{1+2|\gamma_1^L - \gamma_2^L|}, \frac{|\gamma_2^L - \gamma_3^L|}{1+2|\gamma_2^L - \gamma_3^L|}\}, \min\{\frac{|\gamma_1^U - \gamma_2^U|}{1+2|\gamma_1^U - \gamma_2^U|}, \frac{|\gamma_2^U - \gamma_3^U|}{1+2|\gamma_2^U - \gamma_3^U|}\}] \\ &= (\tilde{\mu}_1 O_2 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_2 \tilde{\mu}_3). \end{aligned}$$

□

**Proposition 4.4.** *If  $\tilde{\mu}_1, \tilde{\mu}_2$  and  $\tilde{\mu}_3$  be two interval-valued hesitant fuzzy soft elements. Then the following identities are true:*

- (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) = \tilde{\mu}_1 O_3 \tilde{\mu}_2,$
- (ii)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) = \tilde{\mu}_1 \oplus \tilde{\mu}_2,$
- (iii)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) = \tilde{\mu}_1 O_3 \tilde{\mu}_2,$
- (iv)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) = \tilde{\mu}_1 \otimes \tilde{\mu}_2,$
- (v)  $(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_3 \tilde{\mu}_3 = (\tilde{\mu}_1 O_3 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_3 \tilde{\mu}_3),$
- (vi)  $(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_3 \tilde{\mu}_3 = (\tilde{\mu}_1 O_3 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_3 \tilde{\mu}_3).$

*Proof.* (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_3 \tilde{\mu}_2)$

$$\begin{aligned} &= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}]) \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{2}\}, \min\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{2}\}] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}] \\ &= \tilde{\mu}_1 O_3 \tilde{\mu}_2. \end{aligned}$$

$$\begin{aligned} & \text{(ii)} \quad (\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) \\ &= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}]) \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{2}\}, \max\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{2}\}] \\ &= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \\ &= \tilde{\mu}_1 \oplus \tilde{\mu}_2. \end{aligned}$$

$$\begin{aligned}
& \text{(iii)} \quad (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) \\
&= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}]) \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{2}\}, \min\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{2}\}] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}] \\
&= \tilde{\mu}_1 O_3 \tilde{\mu}_2. \\
& \text{(iv)} \quad (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_3 \tilde{\mu}_2) \\
&= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}]) \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{2}\}, \max\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{2}\}] \\
&= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \\
&= \tilde{\mu}_1 \otimes \tilde{\mu}_2. \\
& \text{(v)} \quad (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_3 \tilde{\mu}_3 \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \max\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_3 \bigcup_{\gamma_3 \in \mu_3} [\tilde{\gamma}_3^L, \tilde{\gamma}_3^U] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\frac{|\max\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}{2}, \frac{|\max\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}{2}] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\max\{\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_2^L - \gamma_3^L|}{2}\}, \max\{\frac{|\gamma_1^U - \gamma_2^U|}{2}, \frac{|\gamma_2^U - \gamma_3^U|}{2}\}] \\
&= (\tilde{\mu}_1 O_3 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_3 \tilde{\mu}_3). \\
& \text{(vi)} \quad (\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_3 \tilde{\mu}_3 \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \min\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_3 \bigcup_{\gamma_3 \in \mu_3} [\tilde{\gamma}_3^L, \tilde{\gamma}_3^U] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\frac{|\min\{\gamma_1^L, \gamma_2^L\} - \gamma_3^L|}{2}, \frac{|\min\{\gamma_1^U, \gamma_2^U\} - \gamma_3^U|}{2}] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2, \gamma_3 \in \mu_3} [\min\{\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_2^L - \gamma_3^L|}{2}\}, \min\{\frac{|\gamma_1^U - \gamma_2^U|}{2}, \frac{|\gamma_2^U - \gamma_3^U|}{2}\}] \\
&= (\tilde{\mu}_1 O_3 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_3 \tilde{\mu}_3). \quad \square
\end{aligned}$$

**Proposition 4.5.** If  $\tilde{\mu}_1, \tilde{\mu}_2$  and  $\tilde{\mu}_3$  be two interval-valued hesitant fuzzy soft elements. Then the following identities are true:

- (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_4 \tilde{\mu}_2) = \tilde{\mu}_1 O_4 \tilde{\mu}_2,$
- (ii)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_4 \tilde{\mu}_2) = \tilde{\mu}_1 \oplus \tilde{\mu}_2,$
- (iii)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_4 \tilde{\mu}_2) = \tilde{\mu}_1 O_4 \tilde{\mu}_2,$
- (iv)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_4 \tilde{\mu}_2) = \tilde{\mu}_1 \otimes \tilde{\mu}_2,$
- (v)  $(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_4 \tilde{\mu}_3 = (\tilde{\mu}_1 O_4 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_4 \tilde{\mu}_3),$
- (vi)  $(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_4 \tilde{\mu}_3 = (\tilde{\mu}_1 O_4 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_4 \tilde{\mu}_3).$

*Proof.* (i)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_4 \tilde{\mu}_2)$

$$\begin{aligned}
&= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}]) \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\min\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{2}\}, \min\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{2}\}] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}] \\
&= \tilde{\mu}_1 O_4 \tilde{\mu}_2.
\end{aligned}$$

(ii)  $(\tilde{\mu}_1 \oplus \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_4 \tilde{\mu}_2)$

$$\begin{aligned}
&= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}]) \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\max\{\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L - \gamma_2^L|}{2}\}, \max\{\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U - \gamma_2^U|}{2}\}] \\
&= \bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \\
&= \tilde{\mu}_1 \oplus \tilde{\mu}_2.
\end{aligned}$$

(iii)  $(\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cap} (\tilde{\mu}_1 O_4 \tilde{\mu}_2)$

$$= (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cap} (\bigcup_{\gamma_1 \in \mu_1, \gamma_2 \in \mu_2} [\frac{|\gamma_1^L - \gamma_2^L|}{2}, \frac{|\gamma_1^U - \gamma_2^U|}{2}])$$

$$\begin{aligned}
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\min\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L \cdot \gamma_2^L|}{2}\}, \min\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U \cdot \gamma_2^U|}{2}\}] \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\frac{|\gamma_1^L \cdot \gamma_2^L|}{2}, \frac{|\gamma_1^U \cdot \gamma_2^U|}{2}] \\
&= \tilde{\mu}_1 O_4 \tilde{\mu}_2.
\end{aligned}$$

$$\begin{aligned}
&\text{(iv)} (\tilde{\mu}_1 \otimes \tilde{\mu}_2) \tilde{\cup} (\tilde{\mu}_1 O_4 \tilde{\mu}_2) \\
&= (\bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \tilde{\cup} (\bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\frac{|\gamma_1^L \cdot \gamma_2^L|}{2}, \frac{|\gamma_1^U \cdot \gamma_2^U|}{2}]) \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\max\{\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \frac{|\gamma_1^L \cdot \gamma_2^L|}{2}\}, \max\{\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U, \frac{|\gamma_1^U \cdot \gamma_2^U|}{2}\}] \\
&= (\bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U]) \\
&= \tilde{\mu}_1 \otimes \tilde{\mu}_2.
\end{aligned}$$

$$\begin{aligned}
&\text{(v)} (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) O_4 \tilde{\mu}_3 \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\max\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \max\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_4 \bigcup_{\gamma_3 \tilde{\in} \mu_3} [\tilde{\gamma}_3^L, \tilde{\gamma}_3^U] \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2, \gamma_3 \tilde{\in} \mu_3} [\frac{|\max\{\gamma_1^L, \gamma_2^L\} * \gamma_3^L|}{2}, \frac{|\max\{\gamma_1^U, \gamma_2^U\} * \gamma_3^U|}{2}] \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2, \gamma_3 \tilde{\in} \mu_3} [\max\{\frac{|\gamma_1^L * \gamma_2^L|}{2}, \frac{|\gamma_2^L * \gamma_3^L|}{2}\}, \max\{\frac{|\gamma_1^U * \gamma_2^U|}{2}, \frac{|\gamma_2^U * \gamma_3^U|}{2}\}] \\
&= (\tilde{\mu}_1 O_4 \tilde{\mu}_3) \tilde{\cup} (\tilde{\mu}_2 O_4 \tilde{\mu}_3).
\end{aligned}$$

$$\begin{aligned}
&\text{(vi)} (\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) O_4 \tilde{\mu}_3 \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2} [\min\{\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\}, \min\{\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\}] O_4 \bigcup_{\gamma_3 \tilde{\in} \mu_3} [\tilde{\gamma}_3^L, \tilde{\gamma}_3^U] \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2, \gamma_3 \tilde{\in} \mu_3} [\frac{|\min\{\gamma_1^L, \gamma_2^L\} * \gamma_3^L|}{2}, \frac{|\min\{\gamma_1^U, \gamma_2^U\} * \gamma_3^U|}{2}] \\
&= \bigcup_{\gamma_1 \tilde{\in} \mu_1, \gamma_2 \tilde{\in} \mu_2, \gamma_3 \tilde{\in} \mu_3} [\min\{\frac{|\gamma_1^L * \gamma_2^L|}{2}, \frac{|\gamma_2^L * \gamma_3^L|}{2}\}, \min\{\frac{|\gamma_1^U * \gamma_2^U|}{2}, \frac{|\gamma_2^U * \gamma_3^U|}{2}\}] \\
&= (\tilde{\mu}_1 O_4 \tilde{\mu}_3) \tilde{\cap} (\tilde{\mu}_2 O_4 \tilde{\mu}_3).
\end{aligned}$$

□

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